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Conformal linear gravity in de Sitter space II

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Abstract From the group theoretical point of view, it is proved that the theory of linear conformal gravity should be written in terms of a tensor field of rank-3 and mixed symmetry (Binegar et al. in Phys. Rev. D 27:2249, 1983). We obtained such a field equation in de Sitter space (Takook et al. in J. Math. Phys. 51:032503, 2010). In this paper, a proper solution to this equation is obtained as a product of a generalized polarization tensor and a massless scalar field and then the conformally invariant two-point function is calculated. This two-point function is de Sitter invariant and free of any pathological large-distance behavior.

1 Introduction

One of the most important goals of theoretical physics is to achieve a proper theory of quantum gravity. Attempts of quantizing General Relativity (GR) have met with some difficulties. Firstly, the principles of general covariance and causality of GR are in conflict with quantum states of conventional quantum field theory since the two principles of GR are closely related to locality but the quantum states are defined globally. Therefore in order to quantize GR, the quantum states or the probability amplitudes must be defined locally in such a case they are compatible with the principles of GR.

Secondly, the gravitational field is long range and seems to travel with the speed of light, thus, its quantum (the graviton, if it exists) should be massless and should propagate on the light cone. Therefore in the first approximation at least, its equation is expected to be conformal invariant. Note that equations of massless particles are conformally invariant (CI),[1] whereas Einstein's equations are not.

Consideration of massless spin-2 particle is of great importance since it is among the central objects in quantum cosmology and quantum gravity. In Minkowski space, the massless field equations are conformally invariant and for every massless representation of Poincaré group there exists only one corresponding representation in the conformal group [8, 9]. Therefore, it seems that we need a theory which remains invariant under conformal transformation (as one expected for massless theories), and this theory should also be invariant under its spacetime symmetry group. (According to group representation theory and its Wigner interpretation in terms of elementary systems, a linear gravitational field should transform according to unitary irreducible representations of its spacetime symmetry group.) Barut and Böhm [8] have shown that for the physical representation of conformal group, the value of conformal Casimir operator is 9. But based on work of Binegar et al. [1], for any tensor field of rank-2, this value becomes 8. Therefore, the tensor field of rank-2 does not correspond to any unitary irreducible representation (UIR) of conformal group. In other words, the tensor field which corresponds to the physical representations of the conformal group must be a tensor field of higher rank.

We extended this group theoretical content to the de Sitter space and used a mixed symmetry tensor field of rank-3 with conformal degree zero, which transforms according to both UIRs of the conformal and de Sitter groups [2]. By mixed symmetry, we mean

$$\Psi_{abc} = -\Psi_{bac}, \quad \sum_{\text{cycl}} \Psi_{abc} = 0,$$

while a field of conformal degree zero satisfies $u^d \partial_d \Psi_{abc} = 0$.

[1]This was first established for the Maxwell equations in [3, 4], for massless spin $1/2$ in [5], and for any spin in [6, 7].

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In this paper, we first obtain the solution to this conformal field equation and then the conformally invariant two-point function is calculated in such a way that de Sitter invariance is preserved and the theory is free of pathological large-distance behavior.

Here, we would like to mention that in the context of linear quantum gravity, it had been proved that the graviton propagator in the linear approximation for largely separated points has a pathological behavior (infrared divergence) and also in de Sitter (dS) background, the dS invariance is broken [10–12]. Some authors have suggested that infrared divergence might lead to instability of the dS space [13, 14]. Accordingly, Tsamis and Woodard have studied a field operator for linear gravity in dS space in terms of flat coordinates [15, 16]. Noting that this coordinate covers only one-half of the dS hyperboloid, they have examined the possibility of quantum instability and have found a quantum field, which breaks dS invariance. However, Antoniadis, Iliopoulos and Tomaras [17] showed that the infrared divergence of the graviton propagator in one loop approximation is gauge dependent, so it should not appear in an effective way as a physical quantity; this later was verified by others [18–21]. Interestingly, it is shown that in indefinite metric field quantization (Krein space method), these two problems are solved. The dS invariance is survived in Krein space setup as long as a Gupta–Bleuler like vacuum is used to calculate the physical graviton two-point function [22–24]. The singularity of the Wightman two-point function (which appears because of the zero mode problem of the Laplace–Beltrami operator on dS space [25]) is removed when prescription of the completely covariant quantization of the minimally coupled scalar field is followed in Krein space [26].

The organization of this paper and its brief outlook are as follows: Sect. 2 is devoted to a brief review of the CI massless spin-2 wave equations in dS space. The solution of the field equation is considered in Sect. 3. It is shown that this solution can be written in terms of a polarization tensor and a massless scalar field as

$$\mathcal{K}_{\alpha\beta}(x)=\mathcal{D}_{\alpha\beta}(x,\partial)\phi(x).$$

In Sect. 4, the CI bi-tensor two-point function $\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x')$ has been calculated in terms of a scalar two-point function, $\mathcal{W}(\mathcal{Z})$. This scalar two-point function plays a central role in obtaining the conformal graviton two-point function, we find this two-point function in indefinite metric field (Krein) quantization method in Sect. 5. Finally a brief conclusion and an outlook for further investigation has been presented. We have supplied some useful mathematical details of calculations in the appendices.

2 De Sitter field equation

Astrophysical data coming from type Ia supernova indicate that our universe is accelerating and can be well approximated by a world with a non-zero positive cosmological constant [27–30]. It means that our universe, in the first approximation, might be in a dS phase. de Sitter space plays an essential role in the inflationary scenario [31] and also its metric becomes important at large-scale universe, since the existence of such non-vanishing positive cosmological constant is proposed to explain the luminosity observations of the farthest supernovas [32, 33]. Thus the quantization of the massless spin-2 field in dS space, without infrared divergence presents an excellent modality for further research and also it can be an important element in our understanding of quantum gravity and quantum cosmology. Let us first review de Sitter space.

2.1 De Sitter space

de Sitter space can be identified by a 4-dimensional hyperboloid embedded in 5-dimensional flat spacetime:

$$X_H=\left\{x\in\mathbb{R}^5;\, x^2=\eta_{\alpha\beta}x^\alpha x^\beta=-H^{-2}=-\frac{3}{\Lambda}\right\},\qquad \alpha,\beta=0,1,2,3,4, \tag{2.1}$$

where $\eta_{\alpha\beta}=\mathrm{diag}(1,-1,-1,-1,-1)$ and H, Λ are the Hubble parameter and cosmological constant, respectively. The dS metric is

$$ds^2=\eta_{\alpha\beta}\,dx^\alpha\,dx^\beta=g^{dS}_{\mu\nu}\,dX^\mu\,dX^\nu,\quad \mu,\nu=0,1,2,3$$

where X^μ's are four spacetime intrinsic coordinates of the dS hyperboloid. Any geometrical object in this space can be written either in terms of four local coordinates X^μ (intrinsic space notation) or five global coordinates x^α (ambient space notation).

Kinematical group of the dS space is the 10-parameter group $SO_0(1,4)$ (connected component of the identity in $O(1,4)$), which is one of the two possible deformations of the Poincaré group. There are two Casimir operators,

$$Q_2^{(1)}=-\frac{1}{2}L^{\alpha\beta}L_{\alpha\beta},\qquad Q_2^{(2)}=-W_\alpha W^\alpha, \tag{2.2}$$

where $W_\alpha=-\frac{1}{8}\epsilon_{\alpha\beta\gamma\sigma\eta}L^{\beta\gamma}L^{\sigma\eta}$, with 10 infinitesimal generators $L_{\alpha\beta}=M_{\alpha\beta}+S_{\alpha\beta}$. The subscript 2 in $Q_2^{(1)}$, $Q_2^{(2)}$ reminds us that the carrier space is constituted by second-rank tensors. $M_{\alpha\beta}$ and $S_{\alpha\beta}$ are the orbital and the spinorial parts, respectively [34]. The symbol $\epsilon_{\alpha\beta\gamma\sigma\eta}$ holds for the usual antisymmetric tensor. “Massless”[2] is used in reference to propagation on the dS light cone (conformal invariance). The conformal invariance and the light-cone propagation, constitute the basis for constructing massless field in dS space. As a matter of fact, we address the massless spin-2 field in dS space, to one kind of representation, namely the lowest representation of rank-2 tensor in discrete series of the dS group. According to the de Sitter group, massless spin-2 field is denoted by $\Pi^{\pm}_{2,2}$ and $\Pi^{\pm}_{2,1}$ in which $\Pi_{p,q}$'s are UIRs of the dS group in its discrete series and the sign $\pm$, stands for the helicity. The pair (p,q) is used to label the UIRs in de Sitter group. It is proved that $\Pi^{\pm}_{2,2}$, have a Minkowskian interpretation.

The compact subgroup of conformal group $SO(2,4)$ is $SO(2)\otimes SO(4)$. Let $C(E;j_1,j_2)$ denote the irreducible projective representation of the conformal group, where E is the eigenvalues of the conformal energy generator of $SO(2)$ and (j_1,j_2) is the $(2j_1+1)(2j_2+1)$ dimensional representation of $SO(4)=SU(2)\otimes SU(2)$. The representation $\Pi^{+}_{2,2}$ has a unique extension to a direct sum of two UIRs $C(3;2,0)$ and $C(-3;2,0)$ of the conformal group, with positive and negative energies, respectively [8, 35]. The latter restricts to the massless Poincaré UIRs $P^{>}(0,2)$ and $P^{<}(0,2)$ with positive and negative energies, respectively. $\mathcal{P}^{\gtrless}(0,2)$ (resp. $\mathcal{P}^{\gtrless}(0,-2)$) are the massless Poincaré UIRs with positive and negative energies and positive (resp. negative) helicity. The following diagrams illustrate these connections:

$$\Pi^{+}_{2,2}\hookrightarrow\begin{array}{c}\mathcal{C}(3,2,0)\\ \oplus\\ \mathcal{C}(-3,2,0)\end{array}\stackrel{H=0}{\longrightarrow}\begin{array}{c}\mathcal{C}(3,2,0)\\ \oplus\\ \mathcal{C}(-3,2,0)\end{array}\begin{array}{c}\hookleftarrow\mathcal{P}^{>}(0,2)\\ \oplus\\ \hookleftarrow\mathcal{P}^{<}(0,2),\end{array}\tag{2.3}$$

$$\Pi^{-}_{2,2}\hookrightarrow\begin{array}{c}\mathcal{C}(3,0,2)\\ \oplus\\ \mathcal{C}(-3,0,2)\end{array}\stackrel{H=0}{\longrightarrow}\begin{array}{c}\mathcal{C}(3,0,2)\\ \oplus\\ \mathcal{C}(-3,0,2)\end{array}\begin{array}{c}\hookleftarrow\mathcal{P}^{>}(0,-2)\\ \oplus\\ \hookleftarrow\mathcal{P}^{<}(0,-2),\end{array}\tag{2.4}$$

where the arrows $\hookrightarrow$ designate unique extension. It is important to note that the representations $\Pi^{\pm}_{2,1}$ do not have corresponding flat limit. Mathematical details of the group contraction and the physical principles underlying the relationship between dS and Poincaré groups can be found in Refs. [35] and [36], respectively.

2.2 Dirac's six-cone formalism and conformal-invariant field equations

The conformal group acts nonlinearly on Minkowski coordinates. Dirac proposed a manifestly conformally covariant formulation in which the Minkowski coordinates are replaced by coordinates on which the conformal group acts linearly. The resultant theory is then formulated on a 5-dimensional hypercone (named Dirac's six-cone) in a 6-dimensional space. This method was first used by Dirac [37] to demonstrate the field equations for spinor and vector fields in $(1+3)$-dimensional spacetime in a manifestly CI form. This approach to conformal symmetry which leads to best path to exploit the physical symmetry was then developed by Mack and Salam [38] and many others [39, 40].

Dirac's six-cone, or Dirac's projection cone, is defined by

$$u^2\equiv\left(u^0\right)^2-\vec{u}^2+\left(u^5\right)^2=\eta_{ab}u^au^b=0,\qquad \eta_{ab}=\mathrm{diag}(1,-1,-1,-1,-1,1),\tag{2.5}$$

where $u^a\in\mathbb{R}^6$, and $\vec{u}\equiv(u^1,u^2,u^3,u^4)$. Reduction to four dimensions is achieved by projection, that is, by fixing the degrees of homogeneity of all fields. Wave equations, subsidiary conditions, etc., must be expressed in terms of operators that are defined intrinsically on the cone. These are well-defined operators that map tensor fields to tensor fields with the same rank on the cone $u^2=0$. So, the resultant equations which are obtained by this method, are conformally invariant.

We studied this method in de Sitter space and obtained the field equations for massless scalar and vector fields [2, 24, 41]. It has been shown that in the flat limit $(H\rightarrow 0)$, these CI equations reduce exactly to their counterpart in Minkowski space, e.g., Maxwell equations are obtained from the vector field case [24, 41]. The mixed symmetric tensor field $F_{\alpha\beta\gamma}$ in dS space is defined for the spin-2 case which is related to the rank-2 field $\mathcal{K}_{\alpha\beta}$ via the following relation:

$$F_{\alpha\beta\gamma}\equiv(\bar{\partial}_\alpha+x_\alpha)\mathcal{K}_{\beta\gamma}-(\bar{\partial}_\beta+x_\beta)\mathcal{K}_{\alpha\gamma},\tag{2.6}$$

where $\bar{\partial}$ is a transverse derivative (Appendix A). For the spin-2 case it is found that [2]

$$(Q_0-2)^2Q_0\mathcal{K}_{\beta\gamma}=0,\quad\text{or equivalently,}\qquad (Q_2+4)^2(Q_2+6)\mathcal{K}_{\alpha\beta}=0,\tag{2.7}$$

where $Q_2\equiv Q_2^{(1)}$, and $Q_0=-\frac{1}{2}M_{\alpha\beta}M^{\alpha\beta}$ (Appendix A).

We like to emphasize that since Dirac's six-cone formalism has been used, Eq. (2.7) is CI, and also leads to the UIR of the dS and conformal groups. At the next stage we will obtain the solution of Eq. (2.7).

3 De Sitter field solution

Let us start with the most generic form of $\mathcal{K}_{\alpha\beta}$ which can be chosen as [42]

$$\mathcal{K}_{\alpha\beta}=\theta_{\alpha\beta}\phi_1+\mathcal{S}\bar{Z}_{1\alpha}K_\beta+D_{2\alpha}K_{g\beta},\tag{3.1}$$

[2] Note that in dS space, concept of mass does not exist by itself as a conserved quantity. The term “massive” is referred to fields that in their zero curvature limit reduce to massive Minkowskian fields [8]. The concept of light-cone propagation, however, does exist and leads to the conformal invariance.

where $\mathcal{S}$ is the symmetrizer operator and Z_1 is a constant 5-dimensional vector, ϕ_1 is a scalar field, K and K_g are two vector fields. Bar over the vector makes it a tangential (or transverse) vector on dS space,

$$\bar{Z}_\alpha = \theta_{\alpha\beta} Z^\beta = Z_\alpha + H^2 x_\alpha x \cdot Z,$$
$$\text{with } x \cdot \bar{Z} \equiv x_\alpha \bar{Z}^\alpha = 0,$$

and $\theta_{\alpha\beta}$ is the transverse projector ($\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta$). The operator D_2 is the generalized gradient defined by (for simplicity from now on we take $H = 1$)

$$D_2 K = \mathcal{S}(\bar{\partial} - x)K.$$

If $\mathcal{K}_{\alpha\beta}$ satisfies the divergenceless and transversality conditions (which are needed in order to relate it to the physical representation) then, after doing some easy algebra, one gets

$$\begin{aligned} &\mathcal{K}' = 0, \qquad x \cdot (K \text{ and } K_g) = \bar{\partial} \cdot K = 0, \\ &2\phi_1 + Z_1 \cdot K + \bar{\partial} \cdot K_g = 0. \end{aligned} \tag{3.2}$$

Substituting $\mathcal{K}_{\alpha\beta}$ in (2.7) results in

$$\begin{cases} (Q_0+4)^2(Q_0+6)\phi_1 \\ \quad = -4\big[Q_1(Q_1+2) \\ \qquad + (Q_0+4)(Q_1+2) \\ \qquad + (Q_0+4)^2\big] Z_1 \cdot K, & \text{(I)} \\ Q_1^2(Q_1+2)K = 0, & \text{(II)} \\ (Q_1+4)^2(Q_1+6)K_g \\ \quad = 2\big[(x \cdot Z_1) Q_1(Q_1+2) \\ \qquad + (Q_1+4)(x \cdot Z_1)(Q_1+2) \\ \qquad + (Q_1+4)^2(x \cdot Z_1)\big] K. & \text{(III)} \end{cases} \tag{3.3}$$

It is easy to show that from relations (3.3-I, 3.3-II) together with the conditions given in Eq. (3.2), one obtains

$$\phi_1 = -\frac{2}{3} Z_1 \cdot K, \qquad Q_0(Q_0-2)^2\phi_1 = 0; \tag{3.4}$$

note that the latter is the massless scalar field equation in dS space [26, 41]. On the other hand, from Eqs. (3.4) and (3.2), we find

$$\bar{\partial} \cdot K_g = \frac{1}{3} Z_1 \cdot K. \tag{3.5}$$

What has been done up to now is to write ϕ_1 and K_g in terms of K, now we want to obtain K.

K is a vector field that satisfies the conditions (3.2), it can be written as [43, 44]

$$K = \bar{Z}_2 \phi_2 + D_1 \phi_3, \tag{3.6}$$

where Z_2 is another 5-dimensional constant vector, ϕ_2 and ϕ_3 are two scalar fields and $D_1 = \bar{\partial}$. Substituting K into (3.3-II) results in

$$Q_0(Q_0-2)^2\phi_2 = 0, \tag{3.7}$$

it is interesting to note that ϕ_2 also satisfies massless field equation. Similarly, ϕ_3 can be written in terms of ϕ_2 as follows (Appendix B):

$$\phi_3 = -\big[(x \cdot Z_2) + (Z_2 \cdot \bar{\partial})\big]\phi_2, \tag{3.8}$$

and from (3.6, 3.4), one obtains

$$\begin{aligned} &K = \big(\bar{Z}_2 - D_1\big[(x \cdot Z_2) + (Z_2 \cdot \bar{\partial})\big]\big)\phi_2, \\ &\phi_1 = -\frac{2}{3} Z_1 \cdot \big[\bar{Z}_2 - D_1\big[(x \cdot Z_2) + (Z_2 \cdot \bar{\partial})\big]\big]\phi_2. \end{aligned} \tag{3.9}$$

From Eq. (3.3-III) and after making use the similar procedure given in Appendix B, it is proved that K_g can be written in terms of K as

$$K_g = \frac{1}{3}\big[4(x \cdot Z_1)K + Z_1 \cdot \bar{\partial} K - x(Z_1 \cdot K)\big], \tag{3.10}$$

note that $x \cdot K_g = 0$ and $\bar{\partial} \cdot K_g = \frac{1}{3} Z_1 \cdot K$.

Gathering all the results and from Eqs. (3.9) and (3.10), we can construct the tensor field $\mathcal{K}_{\alpha\beta}$, in terms of a massless scalar field as follows:

$$\mathcal{K}_{\alpha\beta}(x) = \mathcal{D}_{\alpha\beta}(x, \partial, Z_1, Z_2)\phi_2, \tag{3.11}$$

where $\mathcal{D}$ is the projector tensor defined by

$$\begin{aligned} &\mathcal{D}(x, \partial, Z_1, Z_2) \\ &\quad = \Big[-\frac{2}{3}\theta Z_1 + \mathcal{S}\bar{Z}_1 + \frac{1}{3} D_2\big[4(x \cdot Z_1) + Z_1 \cdot \bar{\partial} - x(Z_1)\big]\Big] \\ &\qquad \times \big[\bar{Z}_2 - D_1\big[(x \cdot Z_2) + (Z_2 \cdot \bar{\partial})\big]\big]. \end{aligned} \tag{3.12}$$

It is more suitable to express tensor field (3.11) in terms of a polarization tensor and de Sitter plane wave and then by taking the flat limit one can fix Z_1 and Z_2. This can be achieved by written ϕ_2 as a de Sitter plane wave [45, 46]

$$\phi_2(x) = (Hx \cdot \xi)^l, \tag{3.13}$$

where $\xi \in \mathbb{R}^5$ lies on the positive null cone $\mathcal{C}^+ = \{\xi \in \mathbb{R}^5; \xi^2 = 0, \xi^0 > 0\}$. Therefore Eq. (3.11) can be easily brought into the following form:

$$\mathcal{K}_{\alpha\beta}(x) = \mathcal{E}_{\alpha\beta}(x, \xi, Z_1, Z_2)(Hx \cdot \xi)^l,$$

where $\mathcal{E}$ is the generalized polarization tensor. Now one can fix the two arbitrary constant vector Z_1 and Z_2 in terms of the polarization tensor of massless spin-2 field in the Minkowskian limit [43]. In which, $l = -1, -2$ and $l = 0, -3$ leads to conformally coupled and minimally coupled massless scalar fields in de Sitter space, respectively [45, 46].

4 Two-point function

The two-point functions in de Sitter space can be written in terms of bi-tensors [47]. These are functions of two points (x, x') which behave like tensors under coordinate transformations at each points. Bi-tensors are called maximally symmetric if they respect de Sitter invariance. Furthermore, as explained in [46] and [43], the axiomatic field theory in de Sitter is based on bi-tensor two-point function. This two-point function is defined by

$$\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x') = \langle\Omega|\mathcal{K}_{\alpha\beta}(x)\mathcal{K}_{\alpha'\beta'}(x')|\Omega\rangle, \tag{4.1}$$

where $x, x' \in X_H$ and $|\Omega\rangle$ is the Fock-vacuum state. The two-point function which is a solution of Eq. (2.7) with respect to x and x', can be written in terms of a scalar two-point function as

$$\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x') = \Delta_{\alpha\beta\alpha'\beta'}\mathcal{W}(x,x'),$$

where $\mathcal{W}(x,x')$ and $\Delta_{\alpha\beta\alpha'\beta'}$, are bi-scalar two-point function and bi-tensor projection operator, respectively.

4.1 Two-point function in ambient space notation

The similar procedure as the previous section is used to obtain the transverse two-point function, therefore one can write

$$\begin{aligned}\mathcal{W}_{\alpha\beta\alpha'\beta'}&(x,x')\\ &= \theta_{\alpha\beta}\theta'_{\alpha'\beta'}\mathcal{W}_0(x,x') + \mathcal{S}\mathcal{S}'\theta_\alpha\cdot\theta'_{\alpha'}W_{1\beta\beta'}(x,x')\\ &\quad + D_{2\alpha}D'_{2\alpha'}W_{g\beta\beta'}(x,x'),\end{aligned} \tag{4.2}$$

note that $D_2D'_2 = D'_2D_2$ and W_1 and W_g are transverse bi-vector two-point functions which will be identified later. At this stage it is shown that the calculation of $\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x')$ could be initiated from x or x', without any difference, this means each choice results to the same equation for $\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x')$. With the choice of x, $\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x')$ must satisfy Eq. (2.7), it is a matter of simple calculation to get the following relations:

$$\begin{cases} (Q_0+4)^2(Q_0+6)\theta'\mathcal{W}_0 \\ \quad = -4\mathcal{S}'\big[Q_1(Q_1+2)+(Q_0+4)(Q_1+2) \\ \qquad + (Q_0+4)^2\big]\theta'\cdot W_1, & \text{(I)} \\ Q_1^2(Q_1+2)W_1 = 0, & \text{(II)} \\ (Q_1+4)^2(Q_1+6)D'_2W_g \\ \quad = 2\mathcal{S}'\big[(x\cdot\theta')Q_1(Q_1+2) \\ \qquad + (Q_1+4)(x\cdot\theta')(Q_1+2) \\ \qquad + (Q_1+4)^2(x\cdot\theta')\big]W_1. & \text{(III)} \end{cases} \tag{4.3}$$

Noting that W_1 is divergenceless, Eq. (4.3-I) implies that

$$\theta'\mathcal{W}_0(x,x') = -\frac{2}{3}\mathcal{S}'\theta'\cdot W_1(x,x'). \tag{4.4}$$

In order to handle Eq. (4.3-II), we write W_1 in terms of two bi-scalar two-point functions as follows:

$$W_1 = \theta\cdot\theta'\mathcal{W}_2 + D_1D'_1\mathcal{W}_3.$$

Substituting W_1 in Eq. (4.3-II) and using the divergenceless condition, one obtains

$$\begin{aligned} &D'_1\mathcal{W}_3 = -\big[x\cdot\theta'\mathcal{W}_2 + \theta'\cdot\bar\partial\mathcal{W}_2\big],\\ &Q_0(Q_0-2)^2\mathcal{W}_2 = 0.\end{aligned} \tag{4.5}$$

In the next section we will consider $\mathcal{W}_2$ in more detail. Setting $\mathcal{W}_2 \equiv \mathcal{W}$, one obtains

$$W_1(x,x') = \big(\theta\cdot\theta' - D_1\big[x\cdot\theta' + \theta'\cdot\bar\partial\big]\big)\mathcal{W}(x,x'), \tag{4.6}$$

then Eq. (4.3-III) leads to

$$D'_2W_g(x,x') = \frac{1}{3}\mathcal{S}'\big[4\big(x\cdot\theta'\big) + \big(\theta'\cdot\bar\partial\big) - x\big(\theta'\big)\big]W_1(x,x'). \tag{4.7}$$

Now, we use Eqs. (4.4), (4.6), and (4.7), to write $\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x')$ as

$$\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x') = \Delta_{\alpha\beta\alpha'\beta'}(x,\partial,x',\partial')\mathcal{W}(x,x'), \tag{4.8}$$

where

$$\begin{aligned}\Delta_{\alpha\beta\alpha'\beta'}&(x,\partial,x',\partial')\\ &= -\frac{2}{3}\mathcal{S}'\theta\theta'\cdot\big(\theta\cdot\theta' - D_1\big[x\cdot\theta'+\theta'\cdot\bar\partial\big]\big)\\ &\quad + \mathcal{S}\mathcal{S}'\theta\cdot\theta'\big(\theta\cdot\theta' - D_1\big[x\cdot\theta'+\theta'\cdot\bar\partial\big]\big)\\ &\quad + \frac{1}{3}D_2\mathcal{S}'\big(4\big(x\cdot\theta'\big) + \big(\theta'\cdot\bar\partial\big) - x\big(\theta'\big)\big)\\ &\quad \times\big(\theta\cdot\theta' - D_1\big[x\cdot\theta'+\theta'\cdot\bar\partial\big]\big).\end{aligned} \tag{4.9}$$

Similarly, with the choice of x', the two-point function (4.2) satisfies Eq. (2.7) (with respect to x', see Appendix A).

In some cases of interest it is useful to express the relations in terms of $\mathcal{Z}$ which is an invariant object under the isometry group $O(1,4)$. It is defined for two given points on the dS hyperboloid x and x', by

$$\mathcal{Z} \equiv -x\cdot x' = 1 + \frac{1}{2}(x-x')^2,$$

note that any function of $\mathcal{Z}$ is dS invariant, as well. It is the work of a few lines to show that (4.8) in terms of $\mathcal{Z}$ becomes

$$\begin{aligned}
&\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x') \\
&\quad = \frac{1}{3}\mathcal{S}\mathcal{S}'\big[\theta_{\alpha\beta}\theta'_{\alpha'\beta'} f_1(\mathcal{Z}) + (\theta_\alpha\cdot\theta'_{\alpha'})(\theta_\beta\cdot\theta'_{\beta'}) f_2(\mathcal{Z}) \\
&\qquad + \theta'_{\alpha'\beta'}(x'\cdot\theta_\alpha)(x'\cdot\theta_\beta) f_3(\mathcal{Z}) \\
&\qquad + (x'\cdot\theta_\alpha)(x'\cdot\theta_\beta)(x\cdot\theta'_{\alpha'})(x\cdot\theta'_{\beta'}) f_4(\mathcal{Z}) \\
&\qquad + (\theta_\alpha\cdot\theta'_{\alpha'})(x\cdot\theta'_{\beta'})(x'\cdot\theta_\beta) f_5(\mathcal{Z}) \\
&\qquad + \theta_{\alpha\beta}(x\cdot\theta'_{\alpha'})(x\cdot\theta'_{\beta'}) f_6(\mathcal{Z})\big]\frac{d}{d\mathcal{Z}}\mathcal{W}(\mathcal{Z}),
\end{aligned} \tag{4.10}$$

in which

$$f_1(\mathcal{Z}) = -\mathcal{Z}\left(1+\mathcal{Z}\frac{d}{d\mathcal{Z}}\right),$$

$$f_2(\mathcal{Z}) = -\mathcal{Z}\left(11+2\mathcal{Z}\frac{d}{d\mathcal{Z}}\right),$$

$$f_3(\mathcal{Z}) = -\left(3+\mathcal{Z}\frac{d}{d\mathcal{Z}}\right)\frac{d}{d\mathcal{Z}},$$

$$f_4(\mathcal{Z}) = -\left(18+10\mathcal{Z}\frac{d}{d\mathcal{Z}}+\mathcal{Z}^2\frac{d^2}{d\mathcal{Z}^2}\right)\frac{d}{d\mathcal{Z}},$$

$$f_5(\mathcal{Z}) = \left(34+31\mathcal{Z}\frac{d}{d\mathcal{Z}}+4\mathcal{Z}^2\frac{d^2}{d\mathcal{Z}^2}\right),$$

$$f_6(\mathcal{Z}) = -\mathcal{Z}\left(15+9\mathcal{Z}\frac{d}{d\mathcal{Z}}+\mathcal{Z}^2\frac{d^2}{d\mathcal{Z}^2}\right).$$

This form of two-point function satisfies the traceless and divergenceless conditions:

$$\bar\partial\cdot\mathcal{W} = \bar\partial'\cdot\mathcal{W} = 0 \quad \text{and}$$
$$\mathcal{W}_{\alpha\beta\alpha'}^{\ \ \ \ \alpha'}(x,x') = \mathcal{W}^{\alpha}_{\ \alpha\alpha'\beta'}(x,x') = 0.$$

4.2 Two-point function in intrinsic space notation

The two-point function (4.10), has been written in ambient space and here we want to project this two-point function to the intrinsic space. It is shown that any maximally symmetric bi-tensor can be expressed as a sum of products of three basic tensors [47] whose coefficients are functions of the geodesic distance and parallel propagator which is defined by

$$n_\mu = \nabla_\mu\sigma(x,x'), \qquad n_{\mu'} = \nabla_{\mu'}\sigma(x,x'),$$
$$g_{\mu\nu'} = -c^{-1}(\mathcal{Z})\nabla_\mu n_{\nu'} + n_\mu n_{\nu'}.$$

The geodesic distance is implicitly defined for $\mathcal{Z} = -x\cdot x'$, by

$$\begin{cases} \mathcal{Z} = \cosh(\sigma), & \text{if } x \text{ and } x' \text{ are time-like separated;} \\ \mathcal{Z} = \cos(\sigma), & \text{if } x \text{ and } x' \text{are space-like separated.} \end{cases} \tag{4.11}$$

The basic bi-tensors in ambient space notation are obtained

$$\bar\partial_\alpha\sigma(x,x'), \qquad \bar\partial'_{\beta'}\sigma(x,x'), \qquad \theta_\alpha\cdot\theta'_{\beta'},$$

which are restricted to the hyperboloid by

$$T_{\mu\nu'} = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x'^{\beta'}}{\partial X'^{\nu'}}T_{\alpha\beta'}.$$

For $\mathcal{Z} = \cos(\sigma)$, one can find

$$n_\mu = \frac{\partial x^\alpha}{\partial X^\mu}\bar\partial_\alpha\sigma(x,x') = \frac{\partial x^\alpha}{\partial X^\mu}\frac{(x'\cdot\theta_\alpha)}{\sqrt{1-\mathcal{Z}^2}},$$

$$n_{\nu'} = \frac{\partial x'^{\beta'}}{\partial X'^{\nu'}}\bar\partial_{\beta'}\sigma(x,x') = \frac{\partial x'^{\beta'}}{\partial X'^{\nu'}}\frac{(x\cdot\theta'_{\beta'})}{\sqrt{1-\mathcal{Z}^2}},$$

$$\begin{aligned}
\nabla_\mu n_{\nu'} &= \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x'^{\beta'}}{\partial X'^{\nu'}}\theta^\varrho_\alpha\theta'^{\gamma'}_{\beta'}\bar\partial_\varrho\bar\partial_{\gamma'}\sigma(x,x') \\
&= c(\mathcal{Z})\left[n_\mu n_{\nu'}\mathcal{Z} - \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x'^{\beta'}}{\partial X^{\nu'}}\theta_\alpha\cdot\theta'_{\beta'}\right],
\end{aligned}$$

where $c^{-1}(\mathcal{Z}) \equiv -\frac{1}{\sqrt{1-\mathcal{Z}^2}}$.

For $\mathcal{Z} = \cosh(\sigma)$, n_μ and n_ν are multiplied by i and then $c(\mathcal{Z})$ becomes $-\frac{i}{\sqrt{1-\mathcal{Z}^2}}$. In both cases we have

$$g_{\mu\nu'} + (\mathcal{Z}-1)n_\mu n_{\nu'} = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x'^{\beta'}}{\partial X'^{\nu'}}\theta_\alpha\cdot\theta'_{\beta'}.$$

Similarly, the two-point functions in ambient space are related to those in de Sitter intrinsic space through

$$Q_{\mu\nu\mu'\nu'} = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x^\beta}{\partial X^\nu}\frac{\partial x'^{\alpha'}}{\partial X'^{\mu'}}\frac{\partial x'^{\beta'}}{\partial X'^{\nu'}}\mathcal{W}_{\alpha\beta\alpha'\beta'}.$$

Finally, the resultant two-point function in dS intrinsic space reads

$$\begin{aligned}
&Q_{\mu\nu\mu'\nu'}(X,X') \\
&\quad = \frac{(1-\mathcal{Z}^2)^2}{3}\mathcal{S}\mathcal{S}'\bigg[g_{\mu\nu}g'_{\mu'\nu'}\frac{f_1}{(1-\mathcal{Z}^2)^2} \\
&\qquad + g_{\mu\mu'}g_{\nu\nu'}\frac{f_2}{(1-\mathcal{Z}^2)^2} \\
&\qquad + g'_{\mu'\nu'}n_\mu n_\nu\frac{f_3}{1-\mathcal{Z}^2} \\
&\qquad + g_{\mu\mu'}n_\nu n_{\nu'}\left(\frac{2(\mathcal{Z}-1)f_2}{(1-\mathcal{Z}^2)^2}+\frac{f_5}{1-\mathcal{Z}^2}\right)
\end{aligned}$$

$$+ n_\mu n_\nu n_{\mu'} n_{\nu'} \left(\frac{f_2}{(1+\mathcal{Z})^2} - \frac{f_5}{1+\mathcal{Z}} + f_4 \right) + g_{\mu\nu} n_{\mu'} n_{\nu'} \frac{f_6}{1-\mathcal{Z}^2} \Bigg] \mathcal{W}(\mathcal{Z}); \tag{4.12}$$

in the next section, $\mathcal{W}(\mathcal{Z})$ is calculated.

5 Scalar field two-point function

In the previous section, $\mathcal{W}_{\alpha\beta\alpha'\beta'}$ was calculated in terms of a scalar field two-point function, $\mathcal{W}(\mathcal{Z})$, that satisfies $Q_0(Q_0-2)^2\mathcal{W}(\mathcal{Z})=0$. Now, we want to obtain this scalar two-point function. Let us write the general statement of $\mathcal{W}(\mathcal{Z})$ as follows:

$$\mathcal{W}(\mathcal{Z}) = c_1 A(\mathcal{Z}) + c_2 B(\mathcal{Z}) + c_3 C(\mathcal{Z}) + c_4 D(\mathcal{Z}) + c_5 E(\mathcal{Z}), \tag{5.1}$$

where c_1, c_2, c_3, c_4, c_5 are constants and each one of $A(\mathcal{Z})$, $B(\mathcal{Z})$, $C(\mathcal{Z})$, $D(\mathcal{Z})$, $E(\mathcal{Z})$ is a part of the answer that satisfies the following equations:

$$Q_0(Q_0-2)^2 A(\mathcal{Z}) = 0, \qquad Q_0(Q_0-2) B(\mathcal{Z}) = 0,$$
$$(Q_0-2)^2 C(\mathcal{Z}) = 0,$$
$$(Q_0-2) D(\mathcal{Z}) = 0, \qquad Q_0 E(\mathcal{Z}) = 0.$$

Each of these functions should be identified. $E(\mathcal{Z})$ can be considered as the two-point function for a minimally coupled massless scalar field in dS space [25, 48]. This two-point function has been found in [48] as follows:

$$E(\mathcal{Z}) = \frac{1}{8\pi^2}\left[\frac{1}{1-\mathcal{Z}} - \ln(1-\mathcal{Z}) + \ln 2 + f\big(\eta,\eta'\big) \right], \tag{5.2}$$

where f is a function of the conformal time η that breaks the dS invariance and because of the term $\ln(1-\mathcal{Z})$, at largely separated points infrared divergence appears. However, in Krein space calculation, one obtains [26, 49]

$$E_K(\mathcal{Z}) = \frac{i}{8\pi^2} \epsilon\big(x^0 - x'^0\big)\big[\delta(1-\mathcal{Z}) + \vartheta(\mathcal{Z}-1)\big], \tag{5.3}$$

where ϑ is the Heaviside step function and

$$\epsilon\big(x^0 - x'^0\big) = \begin{cases} 1 & x^0 > x'^0, \\ 0 & x^0 = x'^0, \\ -1 & x^0 < x'^0. \end{cases} \tag{5.4}$$

Notice that this two-point function has been written in terms of $\mathcal{Z}$, therefore dS invariance is indeed preserved and it is clearly free of infrared divergence. $D(\mathcal{Z})$ is the two-point function for a conformally coupled massless scalar field in dS space [50]:

$$D(\mathcal{Z}) = -\frac{1}{8\pi^2}\left[\frac{1}{1-\mathcal{Z}} - i\pi\epsilon\big(x^0 - x'^0\big)\delta(1-\mathcal{Z}) \right], \tag{5.5}$$

and in the Krein space, we obtain [26]

$$D_K(\mathcal{Z}) = \frac{i}{8\pi} \epsilon\big(x^0 - x'^0\big)\delta(1-\mathcal{Z}). \tag{5.6}$$

It is worth noting that $D(\mathcal{Z})$ preserves the dS invariant in both methods of quantization, either in usual way in Hilbert space or in Krein space quantization method, however, $E(\mathcal{Z})$ is dS invariant only when it is calculated in indefinite metric field quantization method. In other words, in order to have a covariant quantization we should carry out the calculations in Krein space or the Gupta–Bleuler vacuum is needed for quantization [26, 49]. Therefore, we do the calculations in the Krein space (for the sake of simplicity, we omit the index K for a while and write two-point functions in Krein space).

Other functions can be obtained easily by the integration of $E(\mathcal{Z})$ and $D(\mathcal{Z})$:

$$\begin{aligned} A(\mathcal{Z}) = {} & \frac{i}{8\pi^2(1-\mathcal{Z}^2)} \epsilon\big(x^0 - x'^0\big)\Big[-\frac{1}{4}(\mathcal{Z}-1)^2\vartheta(\mathcal{Z}-1) \\ & + \big(\vartheta(\mathcal{Z}-1) + \vartheta(1-\mathcal{Z})\big) \\ & \times \big((\ln|\mathcal{Z}+1| - 1)(\mathcal{Z}+1) \\ & - (\mathcal{Z}-1)\ln 2 - \ln 4 + 2 \big) \Big], \end{aligned} \tag{5.7}$$

$$\begin{aligned} B(\mathcal{Z}) = {} & \frac{i}{8\pi^2(1-\mathcal{Z}^2)} \epsilon\big(x^0 - x'^0\big) \\ & \times \left[\frac{1}{2}(\mathcal{Z}-1)^2\vartheta(\mathcal{Z}-1) + (1-\mathcal{Z})\vartheta(1-\mathcal{Z}) \right], \end{aligned}$$

$$C(\mathcal{Z}) = \frac{i}{8\pi(1-\mathcal{Z}^2)} \epsilon\big(x^0 - x'^0\big)(1-\mathcal{Z})\vartheta(1-\mathcal{Z}).$$

As a result, $\mathcal{W}(\mathcal{Z})$ is obtained in terms of massless minimally and conformally coupled scalar two-point functions.

6 Conclusion

In the framework of quantum field theory, the graviton is supposed to be a mediator of the gravitational field. So, if the graviton exists, it must be a massless spin-2 particle (because the gravitational field has unlimited range and the source of gravitation is a second-rank tensor). On the other hand as proved in footnote 1, the relativistic equations for massless particles are invariant under the conformal transformations.

It was pointed out that Einstein's theory of gravitation, in the linear approximation and in the background field method, $g_{\mu\nu} = g^{BG}_{\mu\nu} + h_{\mu\nu}$, can be considered as a theory of massless symmetric tensor field of rank-2, however, contrary to the Maxwell equations (which in the quantum framework, are regarded as equations of massless spin-1 particle that are conformally invariant), Einstein's equation of gravitation, as well as the equation of $h_{\mu\nu}$, is not conformally invariant. (Notice that conformally invariant equation of the Weyl gravity in its linear form does not transform according to UIRs of the background spacetime symmetry group (for example dS group) [51].) Moreover, there is no successful theory of quantum gravity, since the standard theory of gravity is not renormalizable when quantum gravitational fluctuations are considered. It has been often claimed that the theories whose field equations contain higher order derivatives are better to renormalize than the standard gravity, however, in such theories one should take care about the unitarity [52, 53]. In a series of papers Bender et al., have argued that if such theories are Parity–Time reversal ($\mathcal{PT}$)-symmetric then the unitarity would survive and they have discussed some higher order theories which are both renormalizable and unitary [54–56]. In our case the higher order field equation (2.7) has in fact $\mathcal{PT}$-invariance since the Casimir operators of the de Sitter group are $\mathcal{PT}$-symmetric. On the other hand it is worth noting that equation (2.7) transforms according to the UIRs of de Sitter group. In this work, we solved this field equation and found the proper solution in terms of a generalized polarization tensor and de Sitter plane wave. Then the related CI two-point functions (Eq. (4.10) in ambient space and Eq. (4.12) in dS intrinsic space) were obtained. We would like to emphasize that these two-point functions are invariant under conformal group as well as de Sitter group.

As is well known, in the theory of quantum fields, Green's functions are used to study physical quantities. However, in calculating Green's functions some infinities appear of which most are removed by the means of regularization and renormalization procedures. This theory successfully unified the electromagnetic, weak and strong interactions within the famous so-called Grand Unified Theory. But up to present days, there is no satisfactory quantum description of gravity. One needs such a theory to better understand the influence of the gravitational field on quantum phenomena or to explain some cosmological observations such as the anisotropy of the cosmic microwave background radiation [57]. Therefore, over the past 50 years, one of the great challenges of physics has been the achievement of a proper theory of the quantized gravitational field.

In the previous work, we had found the field equation for the massless spin-2 field in de Sitter space which was conformally invariant [2] and in the present work, the related two-point function was obtained with the following properties: It is invariant under the conformal transformation and is free of infrared divergence. The latter is achieved by carrying out the calculation in Krein space. The result may be important on formulation of the linear quantum gravity in de Sitter space. Actually we believe that quantization in Krein space sheds some light on the problem of the non-renormalizability of quantum gravity.

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Appendix A: Some mathematical preliminaries

In this appendix we first review the Krein space briefly and then collect some useful relations.

Hilbert space is built by a set of modes with positive norms:

$$\mathcal{H} = \left\{ \sum_{k\geq 0} \alpha_k \phi_k ; \sum_{k\geq 0} |\alpha_k|^2 < \infty \right\}, \quad \text{with } (\phi_1, \phi_2) > 0.$$

Krein space is defined as a direct sum of an Hilbert space and an anti-Hilbert space (negative inner product space):

$$\mathcal{K} = \mathcal{H} \oplus \bar{\mathcal{H}},$$

where $\bar{\mathcal{H}}$ stands for the anti-Hilbert space. Note that due to the indefinite inner product space, some states are allowed to have negative norm. These modes are only used as a mathematical tool in renormalization procedure and are ruled out by imposing some conditions. In fact as discussed in [26], in Krein space setup, minimally coupled scalar field is defined on non-Hilbertian Fock space. This is followed by the fact that the one-particle sector is itself not a Hilbert space since the total space (Krein space) is equipped with an indefinite inner product. The physical space is the quotient space: Krein space/negative-norm space. This is a Hilbert space carrying the UIR of the de Sitter group. It is shown that quantization in Krein space either removes some infinities (for example the vacuum energy vanishes without any need of reordering the terms), or at least regularizes the theory (for more details see [58] and references therein).

In what follows, some useful relations that are used in this paper, are listed: $\bar{\partial}_\alpha$ is the tangential (or transverse) derivative on dS space, defined by

$$\bar{\partial}_\alpha = \theta_{\alpha\beta}\partial^\beta = \partial_\alpha + x_\alpha x \cdot \partial, \quad \text{with } x \cdot \bar{\partial} = 0,$$

and also one can define

$$M_{\alpha\beta} \equiv -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) = -i(x_\alpha \bar{\partial}_\beta - x_\beta \bar{\partial}_\alpha),$$
$$S_{\alpha\beta}\mathcal{K}_{\gamma\delta} \equiv -i(\eta_{\alpha\gamma}\mathcal{K}_{\beta\delta} - \eta_{\beta\gamma}\mathcal{K}_{\alpha\delta} + \eta_{\alpha\delta}\mathcal{K}_{\beta\gamma} - \eta_{\beta\delta}\mathcal{K}_{\alpha\gamma}). \tag{A.1}$$

Operator $Q_2^{(1)}$ commutes with the action of the group generators, thus, it is constant in each UIR. The eigenvalues of $Q_2^{(1)}$ can be used to classify the UIR's i.e.,

$$\big(Q_2^{(1)} - \langle Q_2^{(1)}\rangle\big)\mathcal{K}(x) = 0. \tag{A.2}$$

Following Dixmier [59], one can get a classification scheme using a pair (p, q) of parameters involved in the following possible spectral values of the Casimir operators:

$$\begin{aligned} Q_p^{(1)} &= \big(-p(p+1) - (q+1)(q-2)\big)I_d, \\ Q_p^{(2)} &= \big(-p(p+1)q(q-1)\big)I_d. \end{aligned} \tag{A.3}$$

Three types of UIR are distinguished for $SO(1,4)$ according to the range of values of the parameters q and p [59, 60], namely: principal, complementary and discrete series. The flat limit indicates that for the principal and complementary series the value of p bears the meaning of spin. For example in discrete series $p = q = 2$ have a Minkowskian interpretation as a massless spin-2 particle.

The action of the Casimir operators Q_1 and Q_2 can be brought in the more explicit form

$$\begin{aligned} Q_1 K_\alpha &= (Q_0 - 2)K_\alpha + 2x_\alpha \partial \cdot K - 2\partial_\alpha x \cdot K \\ &= (Q_0 - 2)K_\alpha + 2x_\alpha \bar\partial \cdot K - 2\bar\partial_\alpha x \cdot K \\ &\quad + 2x_\alpha(x \cdot K), \end{aligned} \tag{A.4}$$

$$\begin{aligned} Q_2\mathcal{K}_{\alpha\beta} &= (Q_0 - 6)\mathcal{K}_{\alpha\beta} + 2\mathcal{S}x_\alpha \partial \cdot \mathcal{K}_\beta \\ &\quad - 2\mathcal{S}\partial_\alpha x \cdot \mathcal{K}_\beta + 2\eta_{\alpha\beta}\mathcal{K}'. \end{aligned} \tag{A.5}$$

The two-point function (4.8) with the choice of x' reads

$$\mathcal{W}_{\alpha\beta\alpha'\beta'}(x, x') = \Delta'_{\alpha\beta\alpha'\beta'}(x, \partial, x', \partial')\mathcal{W}(x, x'),$$

where

$$\begin{aligned} &\Delta'_{\alpha\beta\alpha'\beta'}(x, \partial, x', \partial') \\ &\quad = -\frac{2}{3}\mathcal{S}\theta'\theta \cdot \big(\theta' \cdot \theta - D'_1[x' \cdot \theta + \theta \cdot \bar\partial']\big) \\ &\qquad + \mathcal{S}\mathcal{S}'\theta \cdot \theta'\big(\theta' \cdot \theta - D'_1[x' \cdot \theta + \theta \cdot \bar\partial']\big) \\ &\qquad + \frac{1}{3}D'_2\mathcal{S}\big(4(x' \cdot \theta) + (\theta \cdot \bar\partial') - x'(\theta)\big) \\ &\qquad \times \big(\theta' \cdot \theta - D'_1[x' \cdot \theta + \theta \cdot \bar\partial']\big), \end{aligned} \tag{A.6}$$

note that the primed operators act only on the primed coordinates.

To obtain the two-point function, the following identities become important:

$$\bar\partial_\alpha f(\mathcal{Z}) = -\big(x' \cdot \theta_\alpha\big)\frac{df(\mathcal{Z})}{d\mathcal{Z}}, \tag{A.7}$$

$$\begin{aligned} &\theta^{\alpha\beta}\theta'_{\alpha\beta} = \theta \cdot \theta' = 3 + \mathcal{Z}^2, \\ &\big(x \cdot \theta'_{\alpha'}\big)\big(x \cdot \theta'^{\alpha'}\big) = \mathcal{Z}^2 - 1, \\ &\big(x \cdot \theta'_\alpha\big)\big(x' \cdot \theta^\alpha\big) = \mathcal{Z}\big(1 - \mathcal{Z}^2\big), \end{aligned} \tag{A.8}$$

$$\begin{aligned} &\bar\partial_\alpha\big(x \cdot \theta'_{\beta'}\big) = \theta_\alpha \cdot \theta'_{\beta'}, \qquad \bar\partial_\alpha\big(x' \cdot \theta_\beta\big) = x_\beta\big(x' \cdot \theta_\alpha\big) - \mathcal{Z}\theta_{\alpha\beta}, \\ &\bar\partial_\alpha\big(\theta_\beta \cdot \theta'_{\beta'}\big) = x_\beta\big(\theta_\alpha \cdot \theta'_{\beta'}\big) + \theta_{\alpha\beta}\big(x \cdot \theta'_{\beta'}\big), \\ &\theta'^{\beta}_{\alpha'}\big(x' \cdot \theta_\beta\big) = -\mathcal{Z}\big(x \cdot \theta'_{\alpha'}\big), \\ &\theta'^{\gamma}_{\alpha'}\big(\theta_\gamma \cdot \theta'_{\beta'}\big) = \theta'_{\alpha'\beta'} + \big(x \cdot \theta'_{\alpha'}\big)\big(x \cdot \theta'_{\beta'}\big), \\ &Q_0 f(\mathcal{Z}) = \big(1 - \mathcal{Z}^2\big)\frac{d^2 f(\mathcal{Z})}{d\mathcal{Z}^2} - 4\mathcal{Z}\frac{df(\mathcal{Z})}{d\mathcal{Z}}, \end{aligned}$$

$$\int \delta(\mathcal{Z})\, d\mathcal{Z} = \vartheta(\mathcal{Z}), \qquad \int \vartheta(\mathcal{Z})\, d\mathcal{Z} = \mathcal{Z}\vartheta(\mathcal{Z}), \tag{A.9}$$

$$\begin{aligned} &\int \frac{\vartheta(\mathcal{Z} - 1)}{\mathcal{Z} + 1}\, d\mathcal{Z} = \vartheta(\mathcal{Z} - 1)\big(\ln|\mathcal{Z} + 1| - \ln 2\big), \\ &\int \vartheta(\mathcal{Z} - 1)\ln|\mathcal{Z} + 1|\, d\mathcal{Z} \\ &\quad = \vartheta(\mathcal{Z} - 1)\big(\ln|\mathcal{Z} + 1| - 1\big)(\mathcal{Z} + 1) \\ &\qquad - \vartheta(\mathcal{Z} - 1)(\ln 4 - 2), \\ &\int \delta'(\mathcal{Z})f(\mathcal{Z})\, d\mathcal{Z} = -\int \delta(\mathcal{Z})f'(\mathcal{Z})\, d\mathcal{Z}. \end{aligned}$$

Appendix B: Mathematical relations underlying Eq. (3.8)

Substituting K in Eq. (3.3-II), yields

$$\begin{aligned} Q_0^2(Q_0 + 2)\phi_3 &= \big(6Q_0^2 + 12Q_0 - 16\big)(x \cdot Z_2)\phi_2 \\ &\quad + (12Q_0 - 16)(Z_2 \cdot \bar\partial)\phi_2, \end{aligned} \tag{B.1}$$

then the general solution for ϕ_3 can be written as

$$\phi_3 = c_1(x \cdot Z_2)\phi_2 + c_2(Z_2 \cdot \bar\partial)\phi_2, \tag{B.2}$$

where c_1 and c_2 are two constants. In order to find c_1 and c_2, we do the following steps:

Step(I): the divergenceless condition ($Q_0\phi_3 = 4x \cdot Z_2\phi_2 + Z_2 \cdot \bar\partial\phi_2$), together with (B.1), results in

$$\begin{aligned} &2(Q_0 + 4)(Q_0 - 2)(x \cdot Z_2)\phi_2 \\ &\quad = (Q_0 - 2)(Q_0 - 8)(Z_2 \cdot \bar\partial)\phi_2, \end{aligned} \tag{B.3}$$

on the other hand from Eq. (3.7) we have

$$\begin{aligned} &(Q_0 - 2)(Q_0 + 4)(Q_0 - 6)(x \cdot Z_2)\phi_2 \\ &\quad = -6(Q_0 - 2)(Q_0 - 4)(Z_2 \cdot \bar\partial)\phi_2, \end{aligned} \tag{B.4}$$

from (B.3) and (B.4), one obtains

$$Q_0(Q_0-2)^2(Z_2\cdot\bar\partial)\phi_2=0,$$
$$Q_0(Q_0-2)^2(Q_0+4)(x\cdot Z_2)\phi_2=0. \tag{B.5}$$

Substituting (B.5) in (B.2) results in

$$Q_0(Q_0-2)^2(Q_0+4)\phi_3=0. \tag{B.6}$$

After doing some straightforward algebra, one obtains the reduced forms of (B.5) and (B.6) as follows:

$$(Q_0-2)^2(Z_2\cdot\bar\partial)\phi_2=0,$$
$$(Q_0-2)^2(Q_0+4)(x\cdot Z_2)\phi_2=0, \tag{B.7}$$

or

$$(Q_0-2)^2(Q_0+4)\phi_3=0. \tag{B.8}$$

Step(II): Using the divergenceless condition and (B.1), one gets

$$Q_0(Q_0+4)(Q_0-2)\phi_3$$
$$=3(Q_0-4)(Q_0-2)(Z_2\cdot\bar\partial)\phi_2. \tag{B.9}$$

Combining (B.9) and (B.4) results in

$$(Q_0+2)(Q_0+4)(Q_0-2)(x\cdot Z_2)\phi_2$$
$$=-2(Q_0+4)(Q_0-2)(Z_2\cdot\bar\partial)\phi_2. \tag{B.10}$$

From (B.3), (B.10), and (B.7), we get

$$Q_0(Q_0+4)(Q_0-2)(x\cdot Z_2)\phi_2$$
$$=-\frac{1}{2}Q_0(Q_0+4)(Q_0-2)(Z_2\cdot\bar\partial)\phi_2, \tag{B.11}$$

and using the divergenceless condition and (B.7), the above equation can be written as

$$Q_0^2(Q_0-2)(Q_0+4)\phi_3$$
$$=-\frac{1}{2}Q_0^2(Q_0-2)(Q_0+4)(Z_2\cdot\bar\partial)\phi_2. \tag{B.12}$$

From Eqs. (B.2) and (B.12), we obtain $c_1-2c_2=1$, and substituting (B.2) into (B.1) together with (B.7) results in $c_1=c_2=-1$.

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